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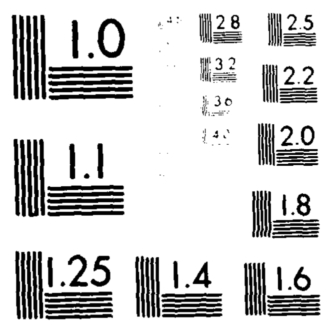

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THE DERIVATION OF A MULTIVARIATE PROBABILITY DENSITY FUNCTION HAVING AN EXPONENTIAL-TYPE BIVARIATE MARGINAL DENSITY

John F. Lennon

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For the calculation of electromagnetic scattering from rough terrain, relations have been established between the scattering cross section and the statistical characterization of the surface. These relations are based on the assignment of a bivariate probability density to the observed surface heights at any two points of a region. To implement a scattering calculation in a region, joint probability density functions that will reduce to appropriate bivariate marginal

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20. Abstract (Continued)

densities are associated with the distribution of heights in the area. In some instances, however, the required form of the multivariate probability density is not evident, since the original form is not necessarily preserved as successive marginal densities are formed. This report describes the procedures used to determine the multivariate probability density function that reduces to the particular bivariate exponential marginal density form having a known correspondence to electromagnetic scattering. Expressions are given both for the multivariate probability density and for the associated marginal densities of any order.

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Preface

The author wishes to thank Dr. Robert J. Papa (RADC/EEC) for helpful conversations that served to clarify several aspects of the problem.

Contents

The Derivation of a Multivariate Probability Density Function Having an Exponential-Type Bivariate Marginal Density

1. BACKGROUND

This report discusses the derivation of a multivariate probability density having certain desired characteristics related to the statistical characterization of terrain features, particularly for use in calculation of the scattering of electromagnetic waves from uneven terrain surfaces. A brief discussion of some surface scattering aspects will serve to place this report in its over-all context. The initial effort in this area is described in some detail in a report by Lennon and Papa.[1] The application of the technique to the actual data base outlined in that report and the subsequent use of the output in electromagnetic scattering calculations have resulted in reasonable agreement with data.[2] The probabilistic elements enter in the representations of terrain cross sections. Ruck et al[3] give expressions for the average bistatic rough surface cross section σ_0 which

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(Received for publication 2 May 1980)

1. Lennon, J. F., and Papa, R.J. (1980) Statistical Characterization of Rough Terrain, RADC-TR-80-9.
2. Papa, R.J., and Lennon, J.F. (1980) Statistical analysis of digitized terrain maps and its application to predicting cross section of rough terrain, Proceedings of the Second Workshop on Terrain and Sea Scatter, George Washington University, Washington, D.C.
3. Ruck, G.T., Barrick, D.C., Stuart, W.D., and Krichbaum, C.K. (1970) Radar Cross Section Handbook, Vol. 2, Plenum Press, New York.

depend on the assumed two-point probability density function (PDF) relating heights of involved surface elements:

$$\sigma_o = |\beta_{pq}|^2 J \ .$$

The height distributions affect J, the factor associated with the contribution from specularly oriented surface slopes. For the case where the surface heights (z_1, z_2) have a bivariate Gaussian relationship,

$$p(z_1, z_2) = \left(\frac{1}{2\pi|\underset{\sim}{R}_2|^{1/2}}\right) e^{-1/2(\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z})} \ ,$$

$$J = \left(\frac{T^2}{\sigma^2 \xi_z^2}\right) \exp\left[-\left(\frac{T^2}{4\sigma^2}\right)\left(\frac{\xi_x^2 + \xi_y^2}{\xi_z^2}\right)\right] \ .$$

Alternatively, if the heights are assumed to share a bivariate exponential form PDF,

$$p(z_1, z_2) = \left(\frac{3}{2\pi|\underset{\sim}{R}_2|^{1/2}}\right) e^{-[3(\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z})]^{1/2}} \ ,$$

then

$$J = \left(\frac{3T^2}{\sigma^2 \xi_z^2}\right) \exp\left[-\left(\frac{\sqrt{6}\,T}{2\sigma}\right)\left(\frac{\xi_x^2 + \xi_y^2}{\xi_z^2}\right)^{1/2}\right]$$

where

σ^2 = variance of the bivariate PDF,

$\underset{\sim}{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ = two dimensional height vector,

$\underset{\sim}{R}_2$ = covariance matrix for the heights,

T = correlation length for the surface,

$\xi_x = \sin\theta_i - \sin\theta_s$,

$\xi_y = 0.0$,

$\xi_z = \cos\theta_i - 1.0$.

The second term of this cross section relation consists of the scattering matrix elements β_{pq}. This becomes

$$\beta_{vv} = \frac{(1 + \cos 2\alpha)\, R_{\parallel}(\alpha)}{(\cos \theta_i + \cos \theta_s)} \quad \text{(vertical polarization)}$$

or,

$$\beta_{hh} = \frac{(1 + \cos 2\alpha)\, R_{\perp}(\alpha)}{(\cos \theta_i + \cos \theta_s)} \quad \text{(horizontal polarization)} \; .$$

In these expressions,

$$R_{\parallel}(\alpha) = \frac{\epsilon_r \cos \alpha - \sqrt{\epsilon_r - \sin^2 \alpha}}{\epsilon_r \cos \alpha + \sqrt{\epsilon_r - \sin^2 \alpha}}$$

and

$$R_{\perp}(\alpha) = \frac{\cos \alpha - \sqrt{\epsilon_r - \sin^2 \alpha}}{\cos \alpha + \sqrt{\epsilon_r - \sin^2 \alpha}}$$

where

θ_i = angle of incidence (with respect to surface normal),

θ_s = angle of scatter (with respect to surface normal),

and

$$\alpha = \left(\frac{\theta_i + \theta_s}{2} \right) \; .$$

Here, ϵ_r is the relative complex dielectric constant of the surface; the subscript $\parallel$ refers to the E-field in the plane of incidence; and the subscript $\perp$ refers to the E-field normal to the plane of incidence. These simplified forms of Ruck's expressions follow from the assumption that the receiver is far from the transmitter so that the portion of the "glistening surface"[4] which contributes to the diffuse multipath is a long, narrow strip extending between the transmitter and

4. Beckmann, P., and Spizzichino, A. (1963) The Scattering of Electromagnetic Waves from Rough Surfaces, Macmillan Co., New York.

receiver. This assumption allows us to make the approximation that the azimuthal scattering angle $\phi_s = 0.0$, which leads to the simplified forms.

The preceding discussion related to a Gaussian two-point-height PDF which is the ordinary case, and an exponential form found to give good agreement for some cases. Both situations, however, require the use of statistical forms describing the relation between heights on a two-point basis. In the initial attempt to describe terrain height relations,[1] the emphasis was on consideration of a multivariate relationship for the surface heights. Each region of interest was divided into a grid structure with ten points in each dimension; the observed heights at these grid points were then considered as a single multivariate observation of an one hundred variate joint PDF. The variates are assumed to have equal means and variances. Determination of the type of multivariate PDF which is more appropriate for the observed data set was made by a hypothesis test.

For the Gaussian case, there is no problem. It is well-known that the form of the Gaussian is preserved, whatever the order of the multivariate PDF. Thus the bivariate marginal density for an N-variate Gaussian (here N = 100) is always a bivariate Gaussian. However, it is not clear what form would be required as the multivariate PDF when an exponential form is the bivariate marginal density. Unlike the Gaussian, the form of the PDF for that case is not preserved for successive marginal densities. This requires some additional explanation.

In that basic report,[1] it is shown that an N-variate exponential PDF would have the form

$$p(z_1, \ldots, z_N) = \left[\frac{(N+1)^{N/2}}{2^{\frac{N+1}{2}} (2\pi)^{\frac{N-1}{2}} \Gamma\left(\frac{N+1}{2}\right) |\underset{\sim}{R}|^{1/2}} \right] e^{-[(N+1)(\underset{\sim}{z}^T \underset{\sim}{R}^{-1} \underset{\sim}{z})]^{1/2}} .$$

Note that this contains the gamma function, $\Gamma(x) \equiv (x-1)!$ It can also be shown that the form for the general L-variate marginal density for this N-variate PDF is (see Appendix A of this report for the derivation)

$$p(z_1, \ldots, z_L) = \left(\frac{N+1}{4}\right)^{\frac{N+L-1}{4}} \left(\frac{2\sqrt{\lambda_1 \ldots \lambda_L}}{\pi^{L/2}\, \Gamma\left(\frac{N+1}{2}\right)} \right) \left([\underset{\sim}{z}^T \underset{\sim}{R}_L^{-1} \underset{\sim}{z}]^{1/2}\right)^{\frac{N-L+1}{2}}$$

$$\times K_{\frac{N-L+1}{2}} \left(\sqrt{N+1}\, [\underset{\sim}{z}^T \underset{\sim}{R}_L^{-1} \underset{\sim}{z}]^{1/2} \right) .$$

Here, $K_\nu(u)$ is the modified Bessel function of the second kind, of order ν. Thus an N-variate exponential PDF corresponds to a bivariate marginal density

$$p(z_1, z_2) = \left(\frac{N+1}{4}\right)^{\frac{N+3}{4}} \left(\frac{2\sqrt{\lambda_1\lambda_2}}{\pi\Gamma\left(\frac{N+1}{2}\right)}\right)\left([\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z}]^{1/2}\right)^{\frac{N-1}{2}}$$

$$\times K_{\frac{N-1}{2}} \left(\sqrt{N+1}\, [\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z}]^{1/2}\right)$$

and not to a simple bivariate exponential PDF. Since the form of the N-variate PDF that would have the exponential bivariate marginal density was not immediately evident, the initial assumption was that the surface heights were either multivariate Gaussian or exponential in distribution. The question of the form for a multivariate PDF which would reduce to the bivariate exponential marginal density was left for future consideration.

In this report, the derivation of the general N-variate PDF having a bivariate exponential marginal density is presented.

2. APPROACH

This section will describe the basic approach used to establish a general form for a multivariate PDF having a bivariate exponential marginal density. The results are based on extrapolation from lower to successively higher order variates until a trend is evident. In the following section, the hypothesized form for the N-variate PDF will be verified.

Prior to the specific analysis, some comments of a general nature should be made. In the original report,[1] it was shown that, for convenience, the correlated form of the quadratic form appearing in the PDF can be replaced by an uncorrelated form through a linear transformation of the variates. A further simplification in variates was introduced which reduces the form to one having unit variance. This procedure will be adopted here, leading to the equivalent relations

$$p(z_1, \ldots, z_N) = c_1\, e^{-[c_2^2 (z_1, \ldots, z_N)^T \underset{\sim}{R}^{-1} (z_1, \ldots, z_N)]^{1/2}}$$

$$p(y_1, \ldots, y_N) = c_1\, e^{-[c_2^2 (\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_N y_N^2]^{1/2}}$$

$$p(w_1, \ldots, w_N) = c_1\, e^{-[c_2^2 (w_1^2 + w_2^2 + \ldots + w_N^2)]^{1/2}}$$

The results are the same for the nonzero mean case. The λ_i values are the eigenvalues of the inverse of the covariance matrix. The value of the quadratic form is unchanged by these transformations. In the determination of the appropriate functional forms (as distinct from the actual calculations of PDF results) the uncorrelated ($\sigma^2 = 1$) form will be considered directly.

One additional point which should be discussed is that, in the above relations and in the remainder of this report, the covariance matrix $\underset{\sim}{R}$ and consequently its inverse are positive definite symmetric. This is a requirement of the properties of probability densities. For the present case, it is satisfied by our assumption that the correlation $\underset{\sim}{\rho}$ has a Gaussian spatial dependence and that the elements of the corresponding covariance matrix, $\sigma_{ij} \epsilon [R_{ij}]$ have the form $\sigma_{ij} = \rho_{ij}\sigma^2$. In the report by Lennon and Papa[1] it was shown that when the elements have that form, the conditions for $\underset{\sim}{R}$ to be positive definite are satisfied. Once $\underset{\sim}{R}$ is positive definite, then so is $\underset{\sim}{R}^{-1}$. This result assures convergence for the distribution integrals.

The initial extrapolation from a bivariate exponential form was the result of manipulations associated with determining coefficients for the N-variate exponential PDF.[1] This involved transformation of coordinates to an (r, θ) space. Consider the function:

$$f(w_1, w_2, w_3, w_4) = c_1(w_1^2 + w_2^2 + w_3^2 + w_4^2)^{-1/2} e^{-c_2\sqrt{w_1^2+w_2^2+w_3^2+w_4^2}} .$$

This is reduced to a function of two variables by integration:

$$f(w_1, w_2) = c_1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (w_1^2 + \dots + w_4^2)^{-1/2} e^{-c_2\sqrt{w_1^2+\dots+w_4^2}} \, dw_3 \, dw_4 .$$

Now let

$$r^2 = w_3^2 + w_4^2 \quad \text{and} \quad a^2 = w_1^2 + w_2^2 .$$

Then

$$f(w_1, w_2) = c_1 \int_0^{\infty} \int_0^{2\pi} \frac{r e^{-c_2(r^2+a^2)^{1/2}}}{\sqrt{r^2 + a^2}} \, dr \, d\theta$$

and setting

$$u = r^2 + a^2 \qquad du = (r^2 + a^2)^{-1/2}\, r\, dr$$

gives

$$f(w_1, w_2) = c_1 \int_0^{2\pi} d\theta \int_{\sqrt{a^2}}^{\infty} e^{-c_2 u}\, du = \left(\frac{2\pi c_1}{c_2}\right) e^{-c_2\sqrt{w_1^2+w_2^2}} \quad .$$

This shows that a function of the assumed type reduces after integration to an exponential function in two variates. Since the major interest in the first terms is the progression to more complex forms, the determination of the coefficients required to produce a PDF from the general functional form at each level will be discussed in Appendix B.

The form that reduces to the bivariate exponential is a combination of a rational function and an exponential term. This four-variate relation suggests the assumption that subsequent forms also consist of combinations of such terms.

The six-variate form is assumed to be

$$f(w_1, w_2, \ldots, w_6) = \left[\frac{c_1' c_2'}{\left(\sqrt{w_1^2 + \ldots + w_6^2}\right)^2} + \frac{c_1'}{\left(\sqrt{w_1^2 + \ldots + w_6^2}\right)^3}\right] \times e^{-c_2'\sqrt{w_1^2+\ldots+w_6^2}} \quad .$$

Then

$$f'(w_1, w_2, w_3, w_4) = c_1' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{c_2'}{\left(\sqrt{w_1^2 + \ldots + w_6^2}\right)^2} + \frac{1}{\left(\sqrt{w_1^2 + \ldots + w_6^2}\right)^3}\right] \times e^{-c_2'\sqrt{w_1^2+\ldots+w_6^2}}\, dw_5\, dw_6$$

or with the same change in variables

$$f'(w_1, w_2, w_3, w_4) = c_1' \int_0^{2\pi} d\theta \int_0^{\infty} \left[\frac{c_2'}{\left(\sqrt{r^2 + a^2}\right)^2} + \frac{1}{\left(\sqrt{r^2 + a^2}\right)^3} \right]$$

$$\times\ e^{-c_2' \sqrt{r^2+a^2}}\ r\, dr$$

where

$$r^2 = w_5^2 + w_6^2 \quad \text{and} \quad a^2 = w_1^2 + w_2^2 + w_3^2 + w_4^2 \ .$$

If we now let $u = \sqrt{r^2 + a^2}$ we have

$$f'(w_1, w_2, w_3, w_4) = 2\pi c_1' \left[\int_{\sqrt{a^2}}^{\infty} c_2'\, u^{-2}\, e^{-c_2' u}\, du + \int_{\sqrt{a^2}}^{\infty} c_2'\, u^{-3}\, e^{-c_2' u}\, du \right] \ ;$$

$$f'(w_1, w_2, w_3, w_4) = 2\pi c_1' c_2' \left[E_i\left(-\, c_2' \sqrt{a^2}\right) + \left(c_2' \sqrt{a^2}\right)^{-1} e^{-c_2' \sqrt{a^2}} \right]$$

$$+\ 2\pi c_1' c_2' \left[-E_i \left(-c_2' \sqrt{a^2} \right) \right] \ ;$$

$$f'(w_1, w_2, w_3, w_4) = \left(\frac{2\pi c_1'}{\sqrt{w_1^2 + w_2^2 + w_3^2 + w_4^2}} \right) e^{-c_2' \sqrt{w_1^2+w_2^2+w_3^2+w_4^2}} \ .$$

This verifies that the assumed form is appropriate. These integral relations and those used below are all found in Gradshteyn and Ryzhik.[5]

Intermediate order forms (P = 3, 5) can now be determined by simple single integrations of the corresponding higher ones (p = 4, 6). As before, consider:

$$f(w_1, w_2, w_3) = \int_{-\infty}^{\infty} \left(\frac{c_1}{\sqrt{w_1^2 + w_2^2 + w_3^2 + w_4^2}} \right) e^{-c_2 \sqrt{w_1^2+w_2^2+w_3^2+w_4^2}}\, dw_4 \ .$$

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5. Gradshteyn, I.S., and Ryzhik, I.M. (1965) Tables of Integrals, Series, and Products, Academic Press, New York.

Then, with $r^2 = w_4^2 + a_2$ and $a^2 = w_1^2 + w_2^2 + w_3^2$

$$f(w_1, w_2, w_3) = 2 \int_{\sqrt{a^2}}^{\infty} \left(\frac{c_1}{\sqrt{r^2 - a^2}} \right) e^{-c_2 r} \, dr$$

or

$$f(w_1, w_2, w_3) = 2c_1 K_o \left(c_2 \sqrt{w_1^2 + w_2^2 + w_3^2} \right)$$

where K_o is the modified Bessel function (MBF) of order zero.

Before addressing the case of $p = 5$, we first note that $f(w_1, w_2, w_3, w_4)$ is also readily expressible as an MBF:

$$f(w_1, w_2, w_3, w_4) = \left(\frac{2c_1^2 c_2}{\pi \sqrt{w_1^2 + w_2^2 + w_3^2 + w_4^2}} \right)^{1/2}$$

$$\times K_{1/2} \left(c_2 \sqrt{w_1^2 + w_2^2 + w_3^2 + w_4^2} \right)$$

and after some manipulation, so is $f(w_1, \ldots, w_6)$:

$$f(w_1, w_2, w_3, w_4, w_5, w_6) = \left[\frac{c_o}{\left(c_2' \sqrt{w_1^2 + \ldots + w_6^2} \right)^{3/2}} \right]$$

$$\times K_{3/2} \left(c_2' \sqrt{w_1^2 + \ldots + w_6^2} \right)$$

(Note, c_o is the constant resulting from the changed form.) Then

$$f(w_1, w_2, w_3, w_4, w_5) = \int_{-\infty}^{\infty} \left[\frac{c_o}{\left(c_2' \sqrt{w_1^2 + \ldots + w_6^2} \right)^{3/2}} \right]$$

$$\times K_{3/2} \left(c_2' \sqrt{w_1^2 + \ldots + w_6^2} \right) dw_6 \quad .$$

The substitutions $a^2 = w_1^2 + \ldots + w_5^2$ and $x = (w_6^2/a^2 + 1)$ give

$$f(w_1, w_2, w_3, w_4, w_5) = \left[(a^2)^{-1/2} (c_2')^{-3/2} c_o\right] \int_1^\infty (x^{-1/2})^{3/2} (x - 1)^{-1/2}$$

$$\times K_{3/2} \left(c_2' \sqrt{a^2} \sqrt{x}\right) dx$$

and

$$f(w_1, w_2, w_3, w_4, w_5) = \left(\frac{c_o \sqrt{2\pi}}{c_2'^2 \sqrt{w_1^2 + \ldots + w_5^2}}\right) K_1 \left(c_2' \sqrt{w_1^2 + \ldots + w_5^2}\right) .$$

At this point, functional forms have been constructed starting with the bivariate exponential and progressing through stages to that for the six-variate case. If specific forms for the corresponding PDF relations (as evaluated in Appendix B) are now introduced, an extrapolation to the general term of the progression can be estimated. Note that the correlated form of the variates is now used:

$$p(z_1, z_2) = \left(\frac{3}{2\pi |\underset{\sim}{R}_2|^{1/2}}\right) e^{-\sqrt{3}(\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z})^{1/2}}$$

$$p(z_1, z_2, z_3) = \left(\frac{3^{3/2}}{2\pi^2 |\underset{\sim}{R}_3|^{1/2}}\right) K_o \left(\sqrt{3} [\underset{\sim}{z}^T \underset{\sim}{R}_3^{-1} \underset{\sim}{z}]^{1/2}\right)$$

$$p(z_1, z_2, z_3, z_4) = \left(\frac{3^2}{2^{3/2} \pi^{5/2} |\underset{\sim}{R}_4|^{1/2}}\right) \left(\sqrt{3} [\underset{\sim}{z}^T \underset{\sim}{R}_4^{-1} \underset{\sim}{z}]^{1/2}\right)^{-1/2}$$

$$\times K_{1/2} \left(\sqrt{3} [\underset{\sim}{z}^T \underset{\sim}{R}_4^{-1} \underset{\sim}{z}]^{1/2}\right)$$

$$p(z_1, z_2, z_3, z_4, z_5) = \left(\frac{3^{5/2}}{2^2 \pi^3 |\underset{\sim}{R}_5|^{1/2}}\right)\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}_5^{-1} \underset{\sim}{z}]^{1/2}\right)^{-1}$$

$$\times K_1\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}_5^{-1} \underset{\sim}{z}]^{1/2}\right)$$

$$p(z_1, z_2, z_3, z_4, z_5, z_6) = \left(\frac{3^3}{2^{5/2} \pi^{7/2} |\underset{\sim}{R}_6|^{1/2}}\right)\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}_6^{-1} \underset{\sim}{z}]^{1/2}\right)^{-3/2}$$

$$\times K_{3/2}\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}_6^{-1} \underset{\sim}{z}]^{1/2}\right) \quad .$$

Based on these forms the hypothesis is advanced that the general form is:

$$p(z_1, z_2, \ldots, z_n) = \left(\frac{3^{N/2}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}} |\underset{\sim}{R}|^{1/2}}\right)\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}^{-1} \underset{\sim}{z}]^{1/2}\right)^{-\left(\frac{N-3}{2}\right)}$$

$$\times K_{\frac{N-3}{2}}\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}^{-1} \underset{\sim}{z}]^{1/2}\right) \; .$$

It now remains to be shown that this satisfies the criteria for an N-variate PDF, and that it has the desired bivariate exponential marginal density.

3. MULTIVARIATE PROBABILITY DENSITY FUNCTION

In this section, the general multivariate form will be shown to satisfy requirements for a probability density function (PDF). As discussed, the covariance matrix is positive definite symmetric and hence the quadratic form $[\underset{\sim}{z}^T \underset{\sim}{R}^{-1} \underset{\sim}{z}]$ is always positive; so is the function. It remains to be shown that the integral of this function is unity.

This involves evaluating the integral:

$$I_o = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(z_1, \ldots, z_N)\, dz_1 \ldots dz_N$$

or

$$I_o = \left(\frac{3^{N/2} |\underset{\sim}{R}|^{1/2} \, 3^{-\left(\frac{N-3}{4}\right)}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}} |\underset{\sim}{R}|^{1/2}} \right) \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{w_1^2 + \ldots + w_N^2}} \right)^{\frac{N-3}{2}}$$

$$\times K_{\frac{N-3}{2}} \left(\sqrt{3} \sqrt{w_1^2 + \ldots + w_N^2} \right) dw_1 \ldots dw_N \quad .$$

Again, for $r = (w_1^2 + \ldots + w_N^2)^{1/2}$ the variables transform to

$$I_o = \left(\frac{3^{\frac{N+3}{4}}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}}} \right) \int_0^{2\pi} d\theta_{N-1} \int_0^{\pi} \sin^{N-2} \theta_{N-2} \, d\theta_{N-2} \cdots$$

$$\cdots \int_0^{\pi} \sin \theta_1 \, d\theta_1 \int_0^{\infty} r^{-\left(\frac{N-3}{2}\right)} K_{\frac{N-3}{2}} (\sqrt{3}\, r) \, r^{N-1} \, dr$$

and thus

$$I_o = \left(\frac{3^{\frac{N+3}{4}} (2\pi)}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}}} \right) \left[\frac{\pi^{\frac{N-2}{2}}}{\Gamma(N/2)} \right] \int_0^{\infty} r^{\frac{N+1}{2}} K_{\frac{N-3}{2}} (\sqrt{3}\, r) \, dr \quad .$$

Then

$$I_o = \left(\frac{3^{\frac{N+3}{4}}}{2^{\frac{N-3}{2}} \sqrt{\pi}\, \Gamma(N/2)} \right) \left[2^{\frac{N-1}{2}} (\sqrt{3})^{-\left(\frac{N+3}{2}\right)} \Gamma(N/2) \Gamma(3/2) \right]$$

and so

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(z_1, \ldots, z_N) \, dz_1 \ldots dz_N = 1$$

as required.

This result confirms that the general multivariate form actually is a valid representation of a PDF.

The demonstration that this PDF does have a bivariate marginal density can be shown directly. However, an approach which leads to a more general result will be employed here. The expression for an arbitrary-order marginal density will be derived and subsequently evaluated for the bivariate marginal case.

The L-variate marginal is given by

$$p(z_1, \ldots, z_L) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{3^{N/2}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}} |\underset{\sim}{R}|^{1/2}} \right) \left(\sqrt{3} [\underset{\sim}{z}^T \underset{\sim}{R}^{-1} \underset{\sim}{z}]^{1/2} \right)^{-\left(\frac{N-3}{2}\right)}$$

$$\times K_{\frac{N-3}{2}} \left(\sqrt{3} [\underset{\sim}{z}^T \underset{\sim}{R}^{-1} \underset{\sim}{z}]^{1/2} \right) dz_N \ldots dz_{L+1} \quad .$$

After coordinate transformation and the substitution

$$s^2 = w_N^2 + w_{N-1}^2 + \ldots + w_{L+1}^2 \quad \text{and} \quad a^2 = w_1^2 + w_2^2 + \ldots + w_L^2$$

$$p(w_1, \ldots, w_L) = \left(\frac{3^{N/2} \sqrt{\lambda_1 \ldots \lambda_L}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}}} \right) \int_0^{2\pi} d\theta_{N-L-1} \int_0^{\pi} \sin^{N-L-2} \theta_{N-L-2} \, d\theta_{N-L-2} \cdots$$

$$\cdots \int_0^{\pi} \sin \theta_1 \, d\theta_1 \int_0^{\infty} \left(\frac{1}{\sqrt{3} \sqrt{s^2 + a^2}} \right)^{\frac{N-3}{2}}$$

$$\times K_{\frac{N-3}{2}} \left(\sqrt{3} \sqrt{s^2 + a^2} \right) s^{N-L-1} \, ds \quad .$$

The introduction of a normalized variable $X = (s^2/a^2 + 1)$ leads to

$$p(w_1, \ldots, w_L) = \left(\frac{3^{N/2} \sqrt{\lambda_1 \ldots \lambda_L}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}}} \right) \left[\frac{2 \pi^{\frac{N-L}{2}}}{\Gamma\left(\frac{N-L}{2}\right)} \right] \left(\frac{a^{\frac{N-2L-3}{2}}}{3^{\frac{N-3}{4}}} \right)$$

$$\times \int_1^{\infty} (x-1)^{\frac{N-L-1}{2}} \left(\frac{1}{\sqrt{x}} \right)^{\frac{N-3}{2}} K_{\frac{N-3}{2}} \left(\sqrt{3a^2} \sqrt{x} \right) \frac{dx}{2\sqrt{x-1}} \quad .$$

After simplification, this becomes

$$p(w_1, \ldots, w_L) = \left(\frac{3^{\frac{N+3}{4}} \sqrt{\lambda_1 \ldots \lambda_L}\, [a^2]^{\frac{N-2L+3}{4}}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}} \Gamma\left(\frac{N-L}{2}\right)} \right)$$

$$\times \int_1^{\infty} (x^{-1/2})^{\frac{N-3}{2}} (x-1)^{\frac{N-L-1}{2}} K_{\frac{N-3}{2}} \left(\sqrt{3a^2}\sqrt{x}\right) dx \quad .$$

Then

$$p(w_1, \ldots, w_L) = \left(\frac{3^{\frac{N+3}{4}} \sqrt{\lambda_1 \ldots \lambda_L}\, [a^2]^{\frac{N-2L+3}{4}}}{2^{\frac{N-1}{2}} \pi^{\frac{N+1}{2}} \Gamma\left(\frac{N-L}{2}\right)} \right) \left[\Gamma\left(\frac{N-L}{2}\right) 2^{\frac{N-L}{2}} \right.$$

$$\left. \times \left(\sqrt{3a^2}\right)^{-\left(\frac{N-L}{2}\right)} K_{\frac{L-3}{2}} \left(\sqrt{3}\ \sqrt{a^2}\right) \right]$$

or

$$p(w_1, \ldots, w_L) = \left(\frac{3^{\frac{L+3}{4}} \sqrt{\lambda_1 \ldots \lambda_L}\, (a^2)^{\frac{3-L}{4}}}{2^{\frac{L-1}{2}} \pi^{\frac{L+1}{2}}} \right) K_{\frac{L-3}{2}} \left(\sqrt{3}\ \sqrt{a^2}\right)$$

and finally,

$$p(z_1, \ldots, z_L) = \left(\frac{3^{L/2} \sqrt{\lambda_1 \ldots \lambda_L}}{2^{\frac{L-1}{2}} \pi^{\frac{L+1}{2}}} \right) \left(\frac{1}{\sqrt{3}\, [\underset{\sim}{z}^T \underset{\sim}{R}_L^{-1} \underset{\sim}{z}]^{1/2}} \right)^{\frac{L-3}{2}}$$

$$\times K_{\frac{L-3}{2}} \left(\sqrt{3}\, [\underset{\sim}{z}^T \underset{\sim}{R}_L^{-1} \underset{\sim}{z}]^{1/2}\right) \quad .$$

This is the expression for the L-variate marginal density of the given PDF. To confirm that it has the desired bivariate form, let L = 2. Then,

$$p(z_1, z_2) = \left(\frac{3\sqrt{\lambda_1 \lambda_2}}{\sqrt{2}\ \pi^{3/2}}\right)\left(\frac{1}{\sqrt{3}\ [\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z}]^{1/2}}\right)^{-1/2} K_{-1/2}\left(\sqrt{3}\ [\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z}]^{1/2}\right) .$$

When the relations $K_{-\mu}(x) = K_{\mu}(x)$ and $K_{1/2}(x) = (\pi/2x)^{1/2}\, e^{-x}$ are introduced, this becomes

$$p(z_1, z_2) = \left(\frac{3\sqrt{\lambda_1 \lambda_2}}{2\pi}\right) e^{-\sqrt{3}[\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z}]^{1/2}} .$$

If we take $|\underset{\sim}{R}_2|^{1/2} = (\lambda_1 \lambda_2)^{-1/2}$, then we can see that this is the desired bivariate marginal form, and the general multivariate PDF does indeed satisfy all the requirements.

4. CONCLUSION

The multivariate probability density function (PDF) which would have a bivariate exponential marginal density has now been determined. At this point, the digitized terrain data can be analyzed with the new multivariate form, replacing the original multivariate exponential PDF in the hypothesis testing procedure. In addition, this PDF can be employed in alternative future terrain characterization techniques. Finally, there still remains the question as to just how much the electromagnetic relations for terrain scattering are affected by the use of this form, as opposed to the multivariate exponential PDF.

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Appendix A

Determination of the L-Variate Marginal Density of a Multivariate Exponential Density

The purpose of this section is to determine the general expression which specifies the marginal density of any order for an N-variate exponential probability density function. The procedures are similar to those of the Appendices A and B of the report by Lennon and Papa.[1]

The multivariate correlated exponential PDF is:

$$p(z_1, \ldots, z_N) = \left(\frac{(N+1)^{N/2}}{2^N \pi^{\frac{N-1}{2}} \Gamma\left(\frac{N+1}{2}\right) |\underset{\sim}{R}|^{1/2}} \right) e^{-\sqrt{N+1}\,[\underset{\sim}{z}^T \underset{\sim}{R}^{-1} \underset{\sim}{z}]^{1/2}} \quad .$$

To obtain the L-variate marginal we integrate over (N - L) variates:

$$p(z_1, \ldots, z_L) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(z_1, \ldots, z_N)\, dz_N\, dz_{N-1} \cdots dz_{L+1} \quad .$$

Then

$$p(w_1, \ldots, w_L) = \left(\frac{(N+1)^{N/2} \sqrt{\lambda_1 \ldots \lambda_L}}{2^N \pi^{\frac{N-1}{2}} \Gamma\left(\frac{N+1}{2}\right)} \right)$$

$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\sqrt{N+1}(w_1^2+w_2^2+\ldots+w_N^2)^{1/2}} dw_N \ldots dw_{L+1} \quad .$$

This represents coordinate transformations to an uncorrelated set of variates with $\sigma^2 = 1$. The integration next uses the Jacobian relation to transform to an $(r, \theta_1, \ldots, \theta_{N-L-1})$ space. Let $r^2 = w_N^2 + w_{N-1}^2 + \ldots + w_{L+1}^2$; then

$$|J(N-L)| = r^{N-L-1} \sin^{N-L-2} \theta_{N-L-2} \sin^{N-L-3} \theta_{N-L-3} \ldots \sin^2 \theta_2 \sin \theta_1$$

and we have

$$p(w_1, \ldots, w_L) = \left(\frac{(N+1)^{N/2} \sqrt{\lambda_1 \ldots \lambda_L}}{2^N \pi^{\frac{N-1}{2}} \Gamma\left(\frac{N+1}{2}\right)} \right)$$

$$\times \int_0^{2\pi} d\theta_{N-L-1} \int_0^{\pi} \sin^{N-L-2} \theta_{N-L-2} \, d\theta_{N-L-2} \cdots$$

$$\cdots \int_0^{\pi} \sin \theta_1 \, d\theta_1 \int_0^{\infty} r^{N-L-1}$$

$$\times e^{-\sqrt{N+1}(r^2+w_L^2+\ldots+w_1^2)^{1/2}} dr \quad .$$

For the purposes of integration we have

$$\int_0^{2\pi} d\theta_{N-L-1} \int_0^{\pi} \sin^{N-L-2} \theta_{N-L-2} \, d\theta_{N-L-2} \cdots \int_0^{\pi} \sin \theta_1 \, d\theta_1 = \frac{2 \pi^{\frac{N-L}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}$$

and

$$\int_0^\infty r^M e^{-\mu(r^2+u^2)^{1/2}} dr = 2^{\nu-1/2} \pi^{-1/2} \mu^{1/2-\nu} u^{\nu+1/2} \Gamma(\nu) K_{\nu+1/2}(\mu u)$$

where

$$\nu = \frac{M+1}{2} \quad .$$

Then for our case

$$u^2 = w_1^2 + w_2^2 + \ldots + w_L^2 \qquad \text{and} \qquad \nu = \left(\frac{N-L}{2}\right)$$

and the result becomes

$$p(w_1, \ldots, w_L) = \left(\frac{(N+1)^{N/2} \sqrt{\lambda_1 \ldots \lambda_L}}{2^N \pi^{\frac{N-1}{2}} \Gamma\left(\frac{N+1}{2}\right)}\right) \left(\frac{2 \pi^{\frac{N-L}{2}}}{\Gamma\left(\frac{N-L}{2}\right)}\right)$$

$$\times \left(\frac{2^{\frac{N-L-1}{2}} \Gamma\left(\frac{N-L}{2}\right)}{\pi^{1/2} (N+1)^{\frac{N-L-1}{4}}}\right) \left[\sqrt{w_1^2 + \ldots + w_L^2}\right]^{\frac{N-L+1}{2}}$$

$$\times K_{\frac{N-L+1}{2}}\left(\sqrt{N+1} \sqrt{w_1^2 + \ldots + w_L^2}\right)$$

which reduces to

$$p(w_1, \ldots, w_L) = \left(\frac{N+1}{4}\right)^{\frac{N+L+1}{4}} \left(\frac{2 \sqrt{\lambda_1 \ldots \lambda_L}}{\pi^{L/2} \Gamma\left(\frac{N+1}{2}\right)}\right)$$

$$\times \left[\sqrt{w_1^2 + \ldots + w_L^2}\right]^{\frac{N-L+1}{2}}$$

$$\times K_{\frac{N-L+1}{2}}\left(\sqrt{N+1} \sqrt{w_1^2 + \ldots + w_L^2}\right)$$

and when transformed to a set of correlated variates

$$p(z_1, \ldots, z_L) = \left(\frac{N+1}{4}\right)^{\frac{N+L+1}{4}} \left(\frac{2\sqrt{\lambda_1 \ldots \lambda_L}}{\pi^{L/2}\,\Gamma\,\frac{N+1}{2}}\right) [(\underset{\sim}{z}^T \underset{\sim}{R}_L^{-1} \underset{\sim}{z})^{1/2}]^{\frac{N-L+1}{2}}$$

$$\times K_{\frac{N-L+1}{2}} \left(\sqrt{N+1}\,(\underset{\sim}{z}^T \underset{\sim}{R}_L^{-1} \underset{\sim}{z})^{1/2}\right) \quad .$$

Appendix B

The Transformation of Functional Forms into Probability Densities

The purpose of this section is to determine values for the coefficients that will satisfy the requirements for a probability density function for the functional forms with successively two through six variates. The procedures are similar to those used in the report by Lennon and Papa[1] involving the zeroth and second moment integrals. In addition, the comments on $\underset{\sim}{R}$ and the modified Bessel function properties also apply in general and will not be discussed for each particular case. For each case, $\underset{\sim}{R}_i$ and its associated eigenvalues (λ) are the appropriate ones for that order.

The bivariate exponential case is known as

$$p(z_1, z_2) = \left(\frac{3}{2\pi\,|\underset{\sim}{R}_2|^{1/2}}\right) e^{-\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}_2^{-1} \underset{\sim}{z}]^{1/2}}$$

The case for N = 3 starts from:

$$f(z_1, z_2, z_3) = c\, K_o\left(b[\underset{\sim}{z}^T \underset{\sim}{R}_3^{-1} \underset{\sim}{z}]^{1/2}\right)$$

Then as before, the first moment is equal to unity or

$$\sqrt{\lambda_1\lambda_2\lambda_3} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c\,K_o\left(b[w_1^2 + w_2^2 + w_3^2]^{1/2}\right) dw_1\,dw_2\,dw_3$$

or, after transformation,

$$\sqrt{\lambda_1\lambda_2\lambda_3} = c\int_0^{2\pi} d\theta_2 \int_0^{\pi} \sin\theta_1\,d\theta_1 \int_0^{\infty} K_o(br)\,r^2\,dr \quad .$$

Similarly, for the second moment

$$\sqrt{\lambda_1\lambda_2\lambda_3}\,(3\sigma^2) = c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{w_1^2}{\lambda_1} + \frac{w_2^2}{\lambda_2} + \frac{w_3^2}{\lambda_3}\right)$$

$$\times K_o\left(b[w_1^2 + w_2^2 + w_3^2]^{1/2}\right) dw_1\,dw_2\,dw_3 \quad .$$

These integrals lead to,

$$\sqrt{\lambda_1\lambda_2\lambda_3} = 2\pi^2 c/b^3 \quad \text{and} \quad \sqrt{\lambda_1\lambda_2\lambda_3} = 6\pi^2 c/b^5$$

respectively, or

$$b = \sqrt{3} \quad \text{and} \quad c = \left(\frac{3^{3/2}}{2\pi^2\,|\underset{\sim}{R}_3|^{1/2}}\right)$$

and finally

$$p(z_1, z_2, z_3) = \left(\frac{3^{3/2}}{2\pi^2\,|\underset{\sim}{R}_3|^{1/2}}\right) K_o\left(\sqrt{3}\,[\underset{\sim}{z}^T\underset{\sim}{R}_3^{-1}\underset{\sim}{z}]^{1/2}\right) \quad .$$

For N = 4, the form is

$$f(z_1, z_2, z_3, z_4) = c\left([\underset{\sim}{z}^T\underset{\sim}{R}_4^{-1}\underset{\sim}{z}]^{1/2}\right)^{-1/2} K_{1/2}\left(b[\underset{\sim}{z}^T\underset{\sim}{R}_4^{-1}\underset{\sim}{z}]^{1/2}\right)$$

The two moment integrals here are

$$\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{w_1^2 + \ldots + w_4^2}}\right)^{1/2}$$

$$\times K_{1/2}\left(b\sqrt{w_1^2 + \ldots + w_4^2}\right) dw_1\, dw_2\, dw_3\, dw_4$$

and

$$\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}\,(4\sigma^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{w_1^2}{\lambda_1} + \ldots + \frac{w_4^2}{\lambda_4}\right)\left(\frac{1}{\sqrt{w_1^2 + \ldots + w_4^2}}\right)^{1/2}$$

$$\times K_{1/2}\left(b\sqrt{w_1^2 + \ldots + w_4^2}\right) dw_1 \ldots dw_4 \quad .$$

Again introducing a change in variables to an $(r, \theta_1, \theta_2, \theta_3)$ space and evaluating the two integrals, we obtain

$$b = \sqrt{3} \quad \text{and} \quad c = \left(\frac{3^{7/4}}{2^{3/2}\,\pi^{5/2}\,|\underset{\sim}{R}_4|^{1/2}}\right)$$

which results in

$$p(z_1, z_2, z_3, z_4) = \left(\frac{3^2}{2^{3/2}\,\pi^{5/2}\,|\underset{\sim}{R}_4|^{1/2}}\right)\left(\sqrt{3}\,[\underset{\sim}{z}^T\underset{\sim}{R}_4^{-1}\underset{\sim}{z}]^{1/2}\right)^{-1/2}$$

$$\times K_{1/2}\left(\sqrt{3}\,[\underset{\sim}{z}^T\underset{\sim}{R}_4^{-1}\underset{\sim}{z}]^{1/2}\right) \quad .$$

For N = 5, the form is

$$f(z_1, z_2, z_3, z_4, z_5) = c\left([\underset{\sim}{z}^T\underset{\sim}{R}_5^{-1}\underset{\sim}{z}]^{1/2}\right)^{-1} K_1\left(b[\underset{\sim}{z}^T\underset{\sim}{R}_5^{-1}\underset{\sim}{z}]^{1/2}\right) \quad .$$

The corresponding moment integrals are

$$\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5} = c \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{w_1^2 + \ldots + w_5^2}}\right)$$

$$\times K_1 \left(b\sqrt{w_1^2 + \cdots + w_5^2}\right) dw_1 \ldots dw_5$$

and

$$\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}\,(5\sigma^2) = c \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{w_1^2}{\lambda_1} + \cdots + \frac{w_5^2}{\lambda_5}\right)\left(\frac{1}{\sqrt{w_1^2 + \ldots + w_5^2}}\right)$$

$$\times K_1\left(b\sqrt{w_1^2 + \cdots + w_5^2}\right) dw_1 \ldots dw_5 \quad .$$

The evaluation of these integrals leads to

$$b = \sqrt{3} \quad \text{and} \quad c = \left(\frac{3^2}{2^2\,\pi^3\,|\underset{\sim}{R}_5|^{1/2}}\right)$$

and

$$p(z_1, z_2, z_3, z_4, z_5) = \left(\frac{3^{5/2}}{2^2\,\pi^3\,|\underset{\sim}{R}_5|^{1/2}}\right)\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}_5^{-1} \underset{\sim}{z}]^{1/2}\right)^{-1} K_1\left(\sqrt{3}\,[\underset{\sim}{z}^T \underset{\sim}{R}_5^{-1} \underset{\sim}{z}]^{1/2}\right) .$$

Finally, for N = 6 the form is

$$f(z_1, z_2, z_3, z_4, z_5, z_6) = c\left([\underset{\sim}{z}^T \underset{\sim}{R}_6^{-1} \underset{\sim}{z}]^{1/2}\right)^{-3/2} K_{3/2}\left(b[\underset{\sim}{z}^T \underset{\sim}{R}_6^{-1} \underset{\sim}{z}]^{1/2}\right) \quad .$$

The moment integrals are

$$\sqrt{\lambda_1\lambda_2 \cdots \lambda_6} = c \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{w_1^2 + \cdots + w_6^2}} \right)^{3/2}$$

$$\times K_{3/2} \left(b \sqrt{w_1^2 + \cdots + w_6^2} \right) dw_1 \cdots dw_6$$

and

$$\sqrt{\lambda_1\lambda_2 \cdots \lambda_6}\,(6\sigma^2) = c \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\frac{w_1^2}{\lambda_1} + \cdots + \frac{w_6^2}{\lambda_6} \right) \left(\frac{1}{\sqrt{w_1^2 + \cdots + w_6^2}} \right)^{3/2}$$

$$\times K_{3/2} \left(b \sqrt{w_1^2 + \cdots + w_6^2} \right) dw_1 \cdots dw_6 \quad .$$

After the same type of evaluation sequence, the results are

$$b = \sqrt{3} \quad \text{and} \quad c = \left(\frac{3^{9/4}}{2^{5/2}\, \pi^{7/2}\, |\underset{\sim}{R}_6|^{1/2}} \right)$$

and

$$p(z_1, z_2, z_3, z_4, z_5, z_6) = \left(\frac{3^3}{2^{5/2}\, \pi^{7/2}\, |\underset{\sim}{R}_6|^{1/2}} \right) \left(\sqrt{3}\, [\underset{\sim}{z}^T \underset{\sim}{R}_6^{-1} \underset{\sim}{z}]^{1/2} \right)^{-3/2}$$

$$\times K_{3/2} \left(\sqrt{3}\, [\underset{\sim}{z}^T \underset{\sim}{R}_6^{-1} \underset{\sim}{z}]^{1/2} \right) \quad .$$

This completes the sequence of successively higher order variate terms used to extrapolate to the general N-variate form of the PDF.

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